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A Combinatorial Outlook on Symmetric Functions

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We establish a combinatorial interpretation for various operations on symmetric functions, such as plethysm, scalar product, and derivation. Thus we obtain proofs of formulas involving symmetric functions in term of combinatorial constructions on permutations.

INTRODUCTION

We will outline in this paper a combinatorial approach for the study of symmetric functions. This will be done in two steps. First, we shall describe the combinatorial context of constructions over permutations, i.e., the theory of *permutation-species* (or simply *S-species*). Then, we will show how to transcribe combinatorial identities about *S*-species into identities for symmetric functions.

Permutation-species have been introduced in [J1] and developped in [B] in order to give a combinatorial approach to series (and their operations) of the form:

$$F(\mathbf{X}) = \sum f_D \mathbf{X}^D/\mathbf{aut}(D) \qquad \text{(summed over all } D\text{'s)}, \tag{1}$$

where $\mathbf{X} = (x_1, x_2, x_3, ...)$, $D = (d_1, d_2, d_3, ...)$ with $\Sigma d_k < \infty$, $\mathbf{X}^D = \prod_{k \geqslant 1} x_k^{d_k}$, $\mathbf{aut}(D) = \prod_{k \geqslant 1} d_k!\ k^{d_k}$, and the f_D's are integers. Such series occur naturally in the study of symmetric functions and in the context of Polya's theory (cycle-index series).

Let us first recall the basic definitions of the theory of S-species. In what follows, **S** stands for the category whose objects are of the form (A, σ) with $\sigma: A \to A$ a permutation of A. Sometimes we simply write σ for (A, σ). Furthermore, an arrow $\theta: (A, \sigma) \to (B, \tau)$ of **S** is a bijection $\theta: A \to B$ such that $\theta\sigma\theta^{-1} = \tau$. Then an **S-species T** is a functor $\mathbf{T}: \mathbf{S} \to \mathbf{E}$ from this category **S** of permutations to the category **E** of finite sets. In the interest of concision, we will often describe such a functor only by its effect on a

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typical permutation (A, σ). For example, we define the **sum**, $\mathbf{T}+\mathbf{M}$, of two S-species $\mathbf{T}$ and $\mathbf{M}$ to be the S-species such that $(\mathbf{T}+\mathbf{M})[\sigma] = \mathbf{T}[\sigma]+\mathbf{M}[\sigma]$ (with $+$ representing disjoint union on the right-hand side). The **cartesian product**, $\mathbf{T}\times\mathbf{M}$, is the S-species such that $(\mathbf{T}\times\mathbf{M})[\sigma] = \mathbf{T}[\sigma]\times\mathbf{M}[\sigma]$, and the **product**, $\mathbf{TM}$, is defined by

$$(\mathbf{TM})[(A, \sigma)] = \sum \mathbf{T}[(A_1, \sigma_1)] \times \mathbf{M}[(A_2, \sigma_2)],$$

where the summation $\sum$ on the right-hand side is to be taken as the disjoint union over the set of all pairs $((A_1, \sigma_1), (A_2, \sigma_2))$, with the A_i's such that $\sigma(A_i) = A_i$ and σ_i is σ restricted to the corresponding A_i. It is clear that both these definitions are functorial in nature. The **derivative of kind n**, $\partial_n(\mathbf{T})$, of an S-species $\mathbf{T}$, is defined by setting $\partial_n\mathbf{T}[(A, \sigma)] = \mathbf{T}[(A+\mathbf{n}, \sigma+\kappa_n)]$, where $\mathbf{n}$ is the set $\{1, 2, ..., n\}$ and κ_n is the cyclic permutation $(1, 2, ..., n)$ of $\mathbf{n}$. To any S-species $\mathbf{T}$ we associate the series

$$\mathbf{T}(\mathbf{X}) = \sum t_D \mathbf{X}^D/\mathbf{aut}(D) \qquad \text{(summed over all } D\text{'s)},$$

where t_D is the cardinality of $\mathbf{T}[\sigma]$. Observe that for any S-species $\mathbf{T}$, the cardinality of the set $\mathbf{T}[\sigma]$ only depends on the cyclic type of the permutation σ. Hence, for $D = (d_1, d_2, d_3, ...)$ such that d_k is the number of cycles of length k in σ $(k = 1, 2, 3, ...)$, the cardinality of $\mathbf{T}[\sigma]$ depends only on D. It has been shown in [B] that

$$\begin{aligned}(\mathbf{T}+\mathbf{M})(\mathbf{X}) &= \mathbf{T}(\mathbf{X}) + \mathbf{M}(\mathbf{X}),\\ (\mathbf{TM})(\mathbf{X}) &= \mathbf{T}(\mathbf{X})\mathbf{M}(\mathbf{X}),\\ (\partial_n \mathbf{T})(\mathbf{X}) &= (n\partial/\partial x_n)\,\mathbf{T}(\mathbf{X}).\end{aligned}$$

Now, let p_k be the **power sum** symmetric functions: $p_k = \sum s_i^k$, where the s_i's constitute a (possibly infinite) set of variables. Then to any S-species $\mathbf{T}$ we can associate a symmetric function $\mathrm{ch}(\mathbf{T})$, the **characteristic** of $\mathbf{T}$, by substitution of the p_k's for the x_k's,

$$\mathrm{ch}(\mathbf{T}) = \sum t_D p^D/\mathbf{aut}(D) \qquad \text{(summed over all } D\text{'s)}, \tag{2}$$

where $p^D = \prod_{k\geqslant 1} p_k^{d}k$. It follows immediately from these definitions that

$$\begin{aligned}\mathrm{ch}(\mathbf{T}+\mathbf{M}) &= \mathrm{ch}(\mathbf{T}) + \mathrm{ch}(\mathbf{M}),\\ \mathrm{ch}(\mathbf{TM}) &= \mathrm{ch}(\mathbf{T})\,\mathrm{ch}(\mathbf{M}),\\ \mathrm{ch}(\partial_n\mathbf{T}) &= (n\partial/\partial p_n)\,\mathrm{ch}(\mathbf{T}).\end{aligned} \tag{3}$$

Hence **ch** is a morphism between the combinatorial algebra of S-species and the algebra of symmetric functions. We will introduce in section one, two other basic operations: the plethysm and scalar product of S-species. Then in Section 2, we shall show that those operations are compatible with the morphism **ch**, so that all combinatorial identities between S-species shall automatically give rise to a corresponding identity in the algebra of symmetric functions via **ch**.

1. Operations on S-Species

Here is one of the most interesting operation on S-species. Let $\mathbf{M}$ be an S-species such that

$$\text{i) } \mathbf{M}[(\mathrm{Id}_{\varnothing}, \varnothing)] = \varnothing,$$

$$\text{ii) } \mathbf{M}[\sigma: (A, \sigma) \to (A, \sigma)] = \mathrm{Id}_{\mathbf{M}[(A,\sigma)]},$$

and let $\mathbf{T}$ be any S-species, we define the **substitution**, $\mathbf{T} \circ \mathbf{M}$, in $\mathbf{T}$ of $\mathbf{M}$, by setting $(\mathbf{T} \circ \mathbf{M})[(A, \sigma)]$ to be the set of all structures that can be obtained in the following manner:

(a) start by choosing any partition π of A compatible with σ (i.e., $\sigma(p)$ is a block of π whenever p is a block of π),

(b) then choose any element of $\mathbf{T}[(\pi, \sigma_\pi)]$, where σ_π is the permutation of π that sends block p into the block $\sigma(p)$,

(c) and finally choose for each block p of π an element m_p of $\mathbf{M}[(p, \sigma_p)]$ such that $\mathbf{M}[\sigma](m_p) = m_{\sigma(p)}$, where σ_p is the permutation of p that sends an element a of p to $\sigma^k(a)$, with k the smallest integer such that $\sigma^k(a)$ is in p.

EXAMPLE 1. Let us define the S-species $\mathbf{C}_k$ to be the S-species characteristic of cycles of length k; i.e., $\mathbf{C}_k[\sigma]$ is the set $\{\sigma\}$ if σ is a cycle of length k, otherwise $\mathbf{C}_k[\sigma]$ is empty. It is clear that the S-species $\mathbf{C}$ characteristic of cycles is equal to $\sum_{k \geqslant 1} \mathbf{C}_k$. One easily verifies that

LEMMA 1. $\mathbf{C}_k \circ \mathbf{C}_n = \mathbf{C}_{kn}$.

Here and from now on, an equality sign "$=$" between S-species will usually mean a natural isomorphism of S-species.

EXAMPLE 2. Other interesting S-species are $\mathbf{U}$ and $\mathbf{e}^{\mathbf{X}}$ such that the set $\mathbf{U}[\sigma]$ is always $\{\sigma\}$; and $\mathbf{e}^{\mathbf{X}}[(A, \sigma)]$ is $\{(A, \sigma)\}$ whenever σ is the identity permutation on A, but is the empty set otherwise. One easily obtains

$\mathbf{U}=\mathbf{e}^{\mathbf{X}}\circ\mathbf{C}$ by the usual observation that any permutation has a unique decomposition in cycles. We will often write $\mathbf{e}^{\mathbf{T}}$ instead of $\mathbf{e}^{\mathbf{X}}\circ\mathbf{T}$. And if $\mathbf{X}^n/n!$ stands for the S-species characteristic of identity permutations on sets of cardinality n, we shall write $\mathbf{T}^n/n!$ for $\mathbf{X}^n/n!\circ\mathbf{T}$.

EXAMPLE 3. For any (set-)species $\mathbf{Q}$, we can construct an S-species Fix($\mathbf{Q}$) by setting:

$$\mathrm{Fix}(\mathbf{Q})[(A,\sigma)]=\{q\mid q\in\mathbf{Q}[A] \text{ and } \mathbf{Q}[\sigma](q)=q\}.$$

For this example, the reader must recall (see [J1]) that a (set-)species $\mathbf{Q}$ is a functor from the category of finite sets (with bijections as arrows) to the category of finite sets (with all functions as arrows). As shown in [B], **Fix** is a functor from the category of all species to the category of S-species that preserves sum, product, and substitution.

EXAMPLE 4. Let $\mathbf{G}$ be any normal subgroup of the symmetric group $\mathbf{S}_n$, we define the S-species $\mathbf{G}$ by setting $\mathbf{G}[(A,\sigma)]$ to be the set $\{(A,\sigma)\}$ if there exist a bijection $f\colon \mathbf{n}\to A$ such that $f\sigma f^{-1}\in\mathbf{G}$, and the empty set in all other cases.

EXAMPLE 5. The S-species **Aut**, of automorphisms, is such that $\mathbf{Aut}[\sigma]$ is the set of all automorphisms (in **S**) from σ to itself. It is well known that the cardinality of the set $\mathbf{Aut}[\sigma]$ is $\mathbf{aut}[D]$ if σ is of cycle-type D.

Remark. The reader can consult [BY] for other examples of interesting S-species. At least let us observe that two S-species $\mathbf{T}_1$ and $\mathbf{T}_2$ need not be isomorphic even if for all permutations σ one has $\#\,\mathbf{T}_1[\sigma]=\#\,\mathbf{T}_2[\sigma]$, where # means cardinality. The diligent reader can convince himself that the S-species $\mathbf{C}^{\bullet}$ of pointed cyclic permutations ($\mathbf{C}^{\bullet}[(A,\sigma(]=A)$ is far from isomorphic to the S-species of cyclic automorphisms $\mathbf{Aut}\times\mathbf{C}$, even if they both give n structures on cyclic permutations of length n.

Let $\lambda=(\lambda_1,\lambda_2,\lambda_3,...,\lambda_q)$ be a partition of n ($0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\cdots\leqslant\lambda_q$, and $\sum_k\lambda_k=n$), and set $D(\lambda)=(d_1,d_2,d_3,...,d_n)$, where d_i is the number of parts of the partition λ equal to i. For any S-species $\mathbf{T}$, we write $\mathbf{T}^{\lambda}$ for the S-species such that $\mathbf{T}^{\lambda}[\sigma]$ is $\mathbf{T}[\sigma]$ if the permutation σ has cycle-type $D(\lambda)$, otherwise $\mathbf{T}^{\lambda}[\sigma]$ is the empty set. We also define $\mathbf{T}_n$ as $\sum_\lambda\mathbf{T}^{\lambda}$, summed over all partitions λ of n. With these conventions, $\mathbf{T}_\lambda$ is then defined by

$$\mathbf{T}_\lambda=\frac{(\mathbf{T}_1)^{d_1}\,(\mathbf{T}_2)^{d_2}\,(\mathbf{T}_3)^{d_3}\cdots(\mathbf{T}_n)^{d_n}}{d_1!\,d_2!\,d_3!\,d_n!}.$$

EXAMPLE 6. With these convections, $\mathbf{U}^{\lambda}$ is the S-species characteristic of permutations of cycle-type λ (more precisely of cycle-type $D(\lambda)=$

$(d_1, d_2, d_3, ..., d_n))$. Also, $\mathbf{U}_n$ is the S-species characteristic of permutations of sets of cardinality n, and $\mathbf{C}_\lambda$ is (up to an isomorphism) the same S-species as $\mathbf{U}^\lambda$. Finally, for the S-species $\mathbf{U}_\lambda$, $\mathbf{U}_\lambda[(A, \sigma)]$ is the set of all set-partitions π of A with d_k parts of cardinality k ($1 \leqslant k \leqslant n$), and such that any part of π is a reunion of cycles of σ.

We finally introduce a new operation on S-species: the **scalar-product** $\langle \mathbf{T}, \mathbf{M} \rangle$. First, we observe that (set-)species act on S-species as scalars. Define the product $\mathbf{QT}$ of an S-species $\mathbf{T}$ by a species $\mathbf{Q}$, to be

$$\mathbf{QT}[(A, \sigma)] = \mathbf{Q}[A] \times \mathbf{T}[(A, \sigma)].$$

With this definition in mind, we set $\langle \mathbf{T}, \mathbf{M} \rangle$ to be the (set-)species that associates to a given finite set A, the set

$$\langle \mathbf{T}, \mathbf{M} \rangle [A] = \sum_\sigma \mathbf{T}[(A, \sigma)] \times \mathbf{M}[(A, \sigma)],$$

where σ runs over the set of all permutations of A. It is clear that $\langle -, - \rangle$ is bilinear.

EXAMPLE 7. Let λ and μ be two distinct partitions of an integer n, then it follows from the definitions that the S-species $\mathbf{U}^\lambda$ and $\mathbf{U}^\mu$ are orthogonal for the scalar-product $\langle -, - \rangle$. Moreover, if A is a set of cardinality n, then

$$\langle \mathbf{U}^\lambda, \mathbf{U}^\lambda \rangle [A] = \{\sigma \mid \sigma \text{ is a permutation of } A \text{ of cycle-type } \lambda\}.$$

We previously observed that $\mathbf{U}^\lambda = \mathbf{C}_\lambda$, thus $\langle \mathbf{U}^\lambda, \mathbf{U}^\mu \rangle = \langle \mathbf{C}_\lambda, \mathbf{C}_\mu \rangle$. We conclude that

PROPOSITION 1. *The family of* S-*species* $(\mathbf{C}_\lambda)$ *is orthogonal, but not orthonormal.*

This family will play a crucial role in our combinatorial interpretation of symmetric functions. It is important to observe that the family $(\mathbf{C}_\lambda)$ is far from being a topological basis of the S-species. This is because the *universe* of S-species is richer than the *universe* of symmetric functions. Before going on with the symmetric function aspect of this work, let us observe that

PROPOSITION 2. *For any* S-*species* $\mathbf{T}$ *and* $\mathbf{M}$, *the functor* $\langle \mathbf{U}, - \rangle : \mathbf{S} \to \mathbf{E}$, *is such that*

(a) $\langle \mathbf{U}, \mathbf{T} \times \mathbf{M} \rangle = \langle \mathbf{U}, \mathbf{T} \rangle \times \langle \mathbf{U}, \mathbf{M} \rangle$

(b) $\langle \mathbf{U}, \mathbf{TM} \rangle = \langle \mathbf{U}, \mathbf{T} \rangle \langle \mathbf{U}, \mathbf{M} \rangle$.

Moreover $\langle \mathbf{U}, - \rangle$ *is a left adjoint of* **Fix**. *This means that for all species* $\mathbf{Q}$

and all S-species **T**, *there is a natural bijection between morphisms from* $\langle \mathbf{U}, \mathbf{T} \rangle$ *to* **Q**, *and those from* **Fix(T)** *to* **Q**.

2. Passage to Symmetric Functions

Let Λ be the ring of symmetric functions (in variables $t_1, t_2, \ldots$). Recall that Λ is a graded ring: we have

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k,$$

where Λ^k consists of the homogeneous symmetric polynomials of degree k. This means that any symmetric function f can be uniquely expressed as $\sum_k f_k$ with the f_k's homogeneous symmetric polynomials of degree k. We remind the reader that any symmetric function can be expressed as a polynomial (with rational coefficients) in the power sums p_n; the plethysm "$*$" of symmetric functions is then characterized (see Macdonald [M]) by the fact that $p_n * p_k = p_{nk}$, and the fact that it is linear and multiplicative.

To any **S**-species **T** we have associated a symmetric function ch(**T**), see (2). One observes readily that $\mathrm{ch}(\mathbf{C}_n) = p_n/n$. We also observed in Lemma 1 that $\mathbf{C}_n \circ \mathbf{C}_k = \mathbf{C}_{nk}$. We evidently conclude that $\mathrm{ch}(\mathbf{C}_n \circ \mathbf{C}_k) = \mathrm{ch}(\mathbf{C}_n) * \mathrm{ch}(\mathbf{C}_k)$. This is a special case of

Proposition 3. *For any* **S**-*species* **T** *and* **M** *such that* $\mathbf{T} \circ \mathbf{M}$ *is well defined, we have*

$$\mathrm{ch}(\mathbf{T} \circ \mathbf{M}) = \mathrm{ch}(\mathbf{T}) * \mathrm{ch}(\mathbf{M}).$$

Outline of Proof. Let $\mathbf{T}_1, \mathbf{T}_2, \mathbf{M}_1$, and $\mathbf{M}_2$ be **S**-species such that: $\# \mathbf{T}_1[\sigma] = \# \mathbf{T}_2[\sigma]$, and $\# \mathbf{M}_1[\sigma] = \# \mathbf{M}_2[\sigma]$, for all permutations σ. Then it is clear that for all **S**-species **T** and **M**,

$$\mathrm{ch}(\mathbf{T}_1 \circ \mathbf{M}) = \mathrm{ch}(\mathbf{T}_2 \circ \mathbf{M}),$$

$$\mathrm{ch}(\mathbf{T} \circ \mathbf{M}_1) = \mathrm{ch}(\mathbf{T} \circ \mathbf{M}_2),$$

whenever these expressions make sense. Thus we can reduce the theorem to the case where **T** and **M** are linear combinations of the $\mathbf{C}_\lambda$'s. But the identies (3) in the Introduction permits us to further reduce to the special case preceding the proposition.

Observations. The symmetric function ch(**U**) is the **complete** symmetric function, and $\mathrm{ch}(\mathbf{U}_n)$ is the n th complete symmetric function. From now on, we write H for ch(**U**), h_n for $\mathrm{ch}(\mathbf{U}_n)$, and P for ch(**C**). Clearly $H = \sum_{n \geq 0} h_n$, $P = \sum_{n \geq 1} p_n/n$, and $H = e^P$ follows from Example 2.

Moreover, we have $\mathrm{ch}(\mathbf{C}_\lambda) = p_\lambda/\mathbf{aut}(\lambda)$, where $\mathbf{aut}(\lambda)$ is $\mathbf{aut}(D(\lambda))$ for short, and p_λ is the symmetric function

$$p_\lambda = p_{\lambda_1} p_{\lambda_1} \cdots p_{\lambda_n}.$$

Hence $h_n = \sum p_\lambda/\mathbf{aut}(\lambda)$, summed over all partitions λ of n. The p_λ's form one of the classical linear bases of the ring of symmetric functions.

We will now define a scalar product on symmetric functions similar to the usual scalar product but under a slightly different guise. It is essentially the "cap product" of Redfield [R] and later of Hall [H]. The modifications introduced in the following definitions are dictated by our outlook, but they reduce to the usual definitions in all interesting cases. Let $s(x)$ be any formal power series

$$s(x) = \sum_{n \geqslant 0} s_n x^n/n!$$

and let $f = \sum_{n \geqslant 0} f_n$ be any symmetric function; we define the product $s(x) f$ of the formal series $s(x)$ and the symmetric function f to be the symmetric function

$$s(x) f = \sum_{n \geqslant 0} s_n f_n.$$

Thus power series become scalars for symmetric functions. With this different oulook in mind, we proceed to define the scalar product $[f, g]$ of symmetric functions $f = \sum_{n \geqslant 0} f_n$ and $g = \sum_{n \geqslant 0} g_n$, to be the formal series

$$[f, g] = \sum_{n \geqslant 0} \langle\langle f_n, g_n \rangle\rangle x^n,$$

where $\langle\langle f_n, g_n \rangle\rangle$ is the usual scalar products on symmetric functions characterized by the fact that the p_λ's form an orthogonal family and the fact that

$$\langle\langle p_\lambda, p_\lambda \rangle\rangle = \mathrm{aut}(\lambda).$$

The interest of this definition lies in

PROPOSITION 4. *For any* **S**-*species* **T** *and* **M**, *we have*

$$\langle \mathbf{T}, \mathbf{M} \rangle(x) = [\mathrm{ch}(\mathbf{T}), \mathrm{ch}(\mathbf{M})],$$

where the (x) *in the left-hand side means passage to the generating series of the species involved.*

Outline of Proof. By an argument similar to the one in Proposition 3, the proof can be reduced to the special case

$$\langle \mathbf{C}_\lambda, \mathbf{C}_\mu \rangle(x) = [\mathrm{ch}(\mathbf{C}_\lambda), \mathrm{ch}(\mathbf{U}_\mu)] \qquad (*)$$

for two partitions λ and μ of n.

But we have already seen that $\mathrm{ch}(\mathbf{C}_\lambda) = p_\lambda/\mathbf{aut}(\lambda)$. Thus the right-hand side of $(*)$ is 0 if λ and μ are different, otherwise $[\mathrm{ch}(\mathbf{C}_\lambda), \mathrm{ch}(\mathbf{C}_\lambda)] = x^n/\mathbf{aut}(\boldsymbol{\lambda})$.

On the other hand, we have also observed that $\langle \mathbf{C}_\lambda, \mathbf{C}_\mu \rangle[A]$ is the empty set whenever λ and μ are distinct partitions, and that for A of cardinality n, $\langle \mathbf{C}_\lambda, \mathbf{C}_\lambda \rangle[A]$ is the set of all permutations of A with cycle-type λ. We conclude that the generating series of $\langle \mathbf{C}_\lambda, \mathbf{C}_\mu \rangle$ is 0 when λ and μ are different, otherwise it is $a_\lambda x^n/n!$, with a_λ equal to the number of permutations of A with cycle-type λ. Since $n!/\mathbf{aut}(\lambda)$ is precisely this number, we have established $(*)$. ∎

Example 8. From the definition, it follows that $\langle \mathbf{U}, \mathbf{U} \rangle$ is isomorphic to the species of permutations. This means that $\langle \mathbf{U}, \mathbf{U} \rangle[A] = \{\sigma \mid \sigma$ is a permutation of $A\}$, and

$$\langle \mathbf{U}, \mathbf{U} \rangle(x) = 1/(1-x).$$

On the other hand, recall that $H = \sum_{n \geqslant 0} h_n = \mathrm{ch}(\mathbf{U})$, with $h_n = \mathrm{ch}(\mathbf{U}_n)$, the homogeneous part of degree n of H, so that

$$[\mathrm{ch}(\mathbf{U}), \mathrm{ch}(\mathbf{U})] = [H, H] = \sum_{n \geqslant 0} \langle\!\langle h_n, h_n \rangle\!\rangle\, x^n.$$

But $\mathbf{U}_n = \sum_\lambda \mathbf{C}_\lambda$, where λ runs over the set of all partitions of n. Hence

$$\begin{aligned}\langle\!\langle h_n, h_n \rangle\!\rangle &= \sum_\lambda \left\langle\!\!\left\langle \frac{p_\lambda}{\mathbf{aut}(\lambda)}, \frac{p_\lambda}{\mathbf{aut}(\lambda)} \right\rangle\!\!\right\rangle \\ &= \sum_\lambda \frac{1}{\mathbf{aut}(\lambda)}.\end{aligned}$$

Proposition 4 implies that $\langle\!\langle h_n, h_n \rangle\!\rangle = 1$, thus we obtain the well-known identity

$$1 = \sum_\lambda \frac{1}{\mathbf{aut}(\lambda)}.$$

Example 9. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n, with $D(\lambda) = (d_1, d_2, d_3, \ldots, d_n)$. If we write $D(\lambda)!$ for

$$D(\lambda)! = d_1!\, d_2! \cdots d_n!$$

then $\mathrm{ch}(\mathbf{U}_\lambda) = h_\lambda / D(\lambda)!$, where

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}.$$

Thus for any partition $\mu = (\mu_1, \mu_2, ..., \mu_k)$ of n, we can give a combinatorial meaning to

$$\langle\langle h_\lambda, p_\mu \rangle\rangle / (D(\lambda)!\,\mathbf{aut}(\mu)) = \text{coef. of } x^n \text{ in } \langle \mathbf{U}_\lambda, \mathbf{U}^\mu \rangle(x).$$

For any fixed set A of cardinality n, it is the number of pairs (σ, π), where σ is a permutation of A of cycle-type μ, and π is a partition of A with d_k parts of cardinality k $(1 \leqslant k \leqslant n)$ and such that any part of π is a reunion of cycles of σ.

In a joint paper (in preparation) with Y-N. Yeh, we shall study the λ-ring structure on S-species characterized by the operators

$$\Psi^n = (\text{---}) \circ \mathbf{C}_n,$$

which satisfy the usual identities for Adam's operators (see [K]). We will also establish the correspondance between the λ-rings of S-species and the λ-ring of symmetric functions.

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